Cosmological bulk viscosity, the Burnett regime, and the BGK equation

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Abstract

Einstein´s field equations in FRW space-times are coupled to the BGK equation in order to derive the stress energy tensor including dissipative effects up to second order in the thermodynamical forces. The space-time is assumed to be matter-dominated, but in a low density regime for which a second order (Burnett) coefficient becomes relevant. Cosmological implications of the solutions, as well as the physical meaning of transport coefficients in an isotropic homogeneous universe are discussed.

1 Introduction

The physics of low density systems is relevant in the description of the late universe. Since a hydrodynamical, Navier-Stokes, description of a low density fluid breaks down for small Knudsen numbers, it is desirable to go beyond this regime and analyze the fluid with the tools that statistical physics

provide as the particle collision frequency decreases. One simple tool useful for this purpose is the BGK equation, which simplifies the formalism of the full Boltzmann equation and provides a good grasp about how the non-equilibrium system evolves. In particular, dissipative effects, and generalized transport coefficients are derivable from the BGK formalism. These coefficients can be incorporated to the stress-energy tensor that is, in turn, coupled to Einstein's field equations.

The relevance of cosmological transport coefficients has already been widely recognized [1], nevertheless, until now, the second order Burnett equations [2] [3] have never been applied to isotropic-homogeneous space-times, although the issue has been addressed in the context of special relativity [4] [5]. One possible motivation for the study of the Burnett-Einstein solutions is the well-known relation between wave scattering processes, density fluctuations and dissipative effects [6] [7]. Dynamic structure factors, possibly relevant for describing CMB anisotropies or distortions, can be derived form the hydrodynamical Burnett approach, as it has already been done for collisionless plasmas [8]-[10]. In this work we analyze the solutions to the BGK equation taking a FRW metric as power series expansions in the relaxation time τ. To second order, the solution is used to obtain the constitutive equations for the stress tensor thus providing explicit expressions for the transport coefficients. The explicit use of these results in specific situations will be deferred for future applications. Here we will be concerned with their physical and general features. Special emphasis will be placed in the significance of a bulk viscosity in homogeneous and isotropic systems. Unfortunate use of language has given rise to an identification of a geometrical aspect of the universe with a conventional dissipative effect. This gives rise to ambiguities that must be clarified. To accomplish this task, the paper is divided as follows: section 2 reviews the BGK equation [11] in FRW space-times and outlines the basic strategy of its solution up to second order by means of a Chapman-Enskog type expansion (Burnett regime). Section three, is dedicated to the analysis of the first and second order distribution functions, the establishment of expressions for the transport coefficients and the construction of the Einstein's field equation with a FRW flat metric. Final remarks about the implications of the Burnett regime in cosmology are included in section four.

2 BGK equation in FRW space-times

We start this section writing the standard form of the BGK equation in its relativistic version for a single component system:

$$\frac{Df}{Dt} = -\frac{f - f^{(0)}}{\tau} \tag{1}$$

where $f = f(x^\mu, v^\mu)$ is the non-equilibrium distribution function, $f^{(0)}$ is the equilibrium function, τ is the collision time (mean free time) and $\frac{D}{Dt}$ is the absolute derivative with respect to time t. The LHS of equation (1) can be written as:

$$\frac{Df}{Dt} = \frac{\partial f}{\partial x^\mu}\, v^\mu + \frac{\partial f}{\partial v^\mu} v^\beta \,(\frac{\partial\, v^\mu}{\partial x^\beta} + v^\alpha \Gamma^\mu_{\alpha\beta}) \tag{2}$$

Assuming a flat, homogeneous and isotropic space-time, Eq. (2) reduces to

$$\frac{Df}{Dt} = \frac{\partial f}{\partial x^4}\, v^4 + \frac{\partial f}{\partial v^4}\, v^\beta\, v^\alpha\, \Gamma^4_{\alpha\beta} \tag{3}$$

Here we wish to make sure that the ensuing consequences remain clear. Eq. (3) has no explicit contributions arising from spatial gradients due to the assumption mentioned above. The second term in the R.H.S. containing the non-vanishing Christoffel symbols has an intrinsic geometrical origin so that all effects arising from it must be associated precisely to such structure. Now, denoting $a(ct)$ the scale factor, c the speed of light, E the total mechanical energy of a single particle of rest mass m_o , γ the usual relativistic factor and $g_{\mu\nu}$ the metric tensor, and taking into account the expressions:

$$g_{\mu\nu} = \begin{bmatrix} a(ct) & 0 & 0 & 0 \\ 0 & a(ct) & 0 & 0 \\ 0 & 0 & a(ct) & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{4}$$

$$x^\mu = \begin{bmatrix} x^1 \\ x^2 \\ x^3 \\ ct \end{bmatrix} \tag{5}$$

$$v^\mu = \begin{bmatrix} u^1\gamma \\ u^2\gamma \\ u^3\gamma \\ \frac{E}{m_o\, c} \end{bmatrix} \tag{6}$$

we can get a simpler form of Eq. (3), namely,

$$\frac{Df}{Dt} = \frac{E}{m_o c^2}\frac{\partial f}{\partial t} + m_o \frac{\partial f}{\partial E} (\frac{1}{a}\frac{da}{dt})(v^1 v^1 + v^2 v^2 + v^3 v^3) \tag{7}$$

or, since $v_\alpha v^\alpha = -c^2$,

$$\frac{Df}{Dt} = \frac{E}{m_o c^2}\frac{\partial f}{\partial t} + m_o \frac{\partial f}{\partial E} (\frac{\theta}{3})(\frac{E^2}{m_o^2 c^2} - c^2) \tag{8}$$

The factor $\theta = \frac{3}{a}\frac{da}{dt}$ is precisely a consequence of the geometrical features appearing in Eq. (3), and it is usually identified with the divergence of the four-velocity. Since the spatial gradients vanish in a FRW space-time, emphasis should me made on the fact that the non-vanishing terms of this divergence come from the Christoffel symbols introduced in the covariant derivative, so that θ has a strictly geometrical origin. The solution of Eq (1), taking into account the expression (8) can be approximated up to second order in τ using the proposal:

$$f = f^{(0)} + \tau f^{(1)} + \tau^2 f^{(2)} + ... \tag{9}$$

Substituting Eq. (9) into Eq. (8) and equating the coefficients of equal powers in constant τ, up to second order, we get:

$$f^{(1)} = -\left[\frac{E}{m_o c^2}\frac{\partial f^{(0)}}{\partial t} + m_o \frac{\partial f^{(0)}}{\partial E}(\frac{\theta}{3})(\frac{E^2}{m_o^2 c^2} - c^2)\right] \tag{10}$$

and

$$f^{(2)} = -\left[\frac{E}{m_o c^2}\frac{\partial f^{(1)}}{\partial t} + m_o \frac{\partial f^{(1)}}{\partial E}(\frac{\theta}{3})(\frac{E^2}{m_o^2 c^2} - c^2)\right] \tag{11}$$

Expressions (10-11) constitute our first and second order corrections to the equilibrium distribution of the BGK equation in a flat FRW space-time. For non-relativistic particles in a FRW metric, Eqs. (10-11) reduce to:

$$f_{NR}^{(1)} = -\left[\frac{\partial f^{(0)}}{\partial t} + m_o u^2 \frac{\partial f^{(0)}}{\partial E}(\frac{\theta}{3})\right] \tag{12}$$

$$f_{NR}^{(2)} = -\left[\frac{\partial f^{(1)}}{\partial t} + m_o u^2 \frac{\partial f^{(1)}}{\partial E}(\frac{\theta}{3})\right] \tag{13}$$

The equilibrium function $f^{(0)}$ is, in this case, the Maxwell-Boltzmann distribution:

$$f^{(0)} = n(\frac{m}{2\pi kT})^{\frac{3}{2}} e^{-\frac{mu^2}{2kT}} = n(\frac{m}{2\pi kT})^{\frac{3}{2}} e^{-\frac{E}{kT}} \tag{14}$$

where n is the particle density, m is the mass of one particle of the simple system, k is Boltzmann's constant and T is the temperature.

If particles are conserved, then the continuity equation holds:

$$\frac{\partial n}{\partial t} + n\theta = 0 \tag{15}$$

and the first term at the RHS of Eq. (12) can be written as:

$$\frac{\partial f^{(0)}}{\partial t} = \frac{\partial f^{(0)}}{\partial n}\frac{\partial n}{\partial t} = -\theta f^{(0)} \tag{16}$$

In Eq. (16) θ, usually identified with $\nabla \cdot \mathbf{u}$ has, as emphasized before, a purely geometrical origin; it is, indeed the surviving term of the covariant derivative of a four velocity $v^{\alpha}_{;\alpha}$. Straightforward calculations can now be carried, so that:

$$f^{(1)}_{NR} = (1 + \frac{mu^2}{3kT})\theta f^{(0)} \tag{17}$$

$$f^{(2)}_{NR} = \left[-(1 + \frac{5mu^2}{9kT} - \frac{2}{27}\frac{m^2u^4}{k^2T^2})\theta^2 + \theta\frac{d\theta}{dt} \right] f^{(0)} \tag{18}$$

These corrections obviously vanish for homogeneous, isotropic, non-expanding space-times. The thermodynamical force inherent to the deviations from the perfect distribution is, to first order, the expansion scalar θ. To second order (Burnett regime), the generalized thermodynamical forces are, as expected, non-linear forces appearing in the linear constitutive relations. Here, due to the assumption on space-time, they turn out to be simply θ^2 and $\theta\frac{d\theta}{dt}$. The effect of these second order expressions on the field equations will be the subject of the next section.

3 Stress-tensor and field equation

The Einstein field equations with vanishing cosmological constant are given by:

$$G^{\mu}_{\nu} = \kappa T^{\mu}_{\nu} \tag{19}$$

where G^μ_ν is the Einstein tensor, κ is the coupling constant and T^μ_ν is the stress-energy tensor. The tensor T^μ_ν is related to the non-equilibrium distribution function, up to second order, by:

$$T^\mu_\nu \simeq m \int f^{(0)} v^\mu v_\nu \, dV + m\tau \int f^{(1)} v^\mu v_\nu \, dV + m\tau^2 \int f^{(2)} v^\mu v_\nu \, dV \tag{20}$$

Eq. (20) is simply the ordinary stress-energy tensor written in terms of averages of the molecular velocities. The hydrodynamical velocity vanishes in the comoving frame. In this case, the first term of the R.H.S. of Eq. (20) may be written as:

$$\overset{(0)}{T^\mu_\nu} = \frac{4\pi}{3} m \, h^\mu_\nu \int f^{(0)} u^4 \, du + \rho U^\mu U_\nu = \rho U^\mu U_\nu + P(\delta^\mu_\nu + \frac{U^\mu U_\nu}{c^2}) = \rho U^\mu U_\nu + P h^\mu_\nu \tag{21}$$

where ρ is the mass-energy density, $P = nkT$ is the pressure, $h^\mu_\nu = \delta^\mu_\nu + \frac{U^\mu U_\nu}{c^2}$, and U^μ is the hydrodynamic velocity given by:

$$U^\mu = \begin{bmatrix} 0 \\ 0 \\ 0 \\ c \end{bmatrix}, \; U_\mu = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -c \end{bmatrix} \tag{22}$$

The RHS of Eq. (21) corresponds to the stress tensor of a non-dissipative fluid. If we wish to include dissipation up to second order in τ for non-relativistic particles, we can use expressions (17-18) . Using this information, and assuming that no dissipation in the time axes occur, we can write for the first order correction of the stress energy tensor:

$$\overset{(1)}{T^\mu_\nu} = -\left[4\pi\tau m h^\mu_\nu \int_o^\infty \left(1 + \frac{1}{3}\frac{m_o u^2}{kT}\right) f^{(0)} \, u^4 du\right] \theta = -\eta_c \theta \, h^\mu_\nu \tag{23}$$

Following the rules of linear irreversible thermodynamics [13], the relationship between the induced current $\overset{(1)}{T^\mu_\nu}$ and the force θ which again we emphasize arises from the inherent geometrical aspect of space-time, is $\overset{(1)}{T^\mu_\nu} = -\eta_c \theta h^\mu_\nu$. η_c, the transport coefficient is usually referred to in the literature as bulk viscosity. This is misleading, since θ does not contain spatial gradients. For

this reason, we shall refer to it as the cosmological viscosity η_c, here explicitly given by:

$$\eta_c = 4\pi\tau m \int_o^\infty \left(1 + \frac{1}{3}\frac{m_o u^2}{kT}\right) f^{(0)}\, u^4 du \tag{24}$$

The second order contributions to the stress tensor can now be identified as:

$$\begin{aligned}\overset{(2)}{T^\mu_\nu} \;=\; & -\left[4\pi\tau^2 m h^\mu_\nu \int_o^\infty \left(1 + \frac{5}{9}\frac{m_o u^2}{kT} + \frac{2}{27}\frac{m_o^2 u^4}{k^2 T^2}\right) f^{(0)}\, u^4 du\right]\theta^2 + \\ & \left[4\pi\tau^2 m h^\mu_\nu \int_o^\infty \left(1 + \frac{1}{3}\frac{m_o u^2}{kT}\right) f^{(0)}\, u^4 du\right]\theta\frac{d\theta}{dt}\end{aligned} \tag{25}$$

Thus, two Burnett transport coefficients arise, one for each generalized thermodynamical force θ^2, $\theta\frac{d\theta}{dt}$, namely:

$$B_1 = \left[4\pi\tau^2 m \int_o^\infty \left(1 + \frac{5}{9}\frac{m_o u^2}{kT} + \frac{2}{27}\frac{m_o^2 u^4}{k^2 T^2}\right) f^{(0)}\, u^4 du\right] \tag{26}$$

and

$$B_2 = \left[4\pi\tau^2 m \int_o^\infty \left(1 + \frac{1}{3}\frac{m_o u^2}{kT}\right) f^{(0)}\, u^4 du\right] \tag{27}$$

and the field equations (19), in the Burnett regime, in flat FRW space-times, take the rather interesting form

$$\frac{2}{3}\frac{d\theta}{dt} + \frac{5}{27}\theta^2 = \kappa\left(p - \eta_c\theta - B_1\theta^2 + B_2\theta\frac{d\theta}{dt}\right) \tag{28}$$

$$\frac{1}{3}\theta^2 = \kappa\rho c^2 \tag{29}$$

These equations, clearly dependent on the mean free time τ, yield the dynamics of simple fluid consisting of non-relativistic particles in flat FRW metric in the Burnett regime. Future work will be dedicated to the study of their solutions.

The order of magnitude of these coefficients is readily calculated. Substituting $f^{(0)}$ one obtains that

$$\eta_c \approx 2nkT\tau = \frac{B_1}{\tau} \tag{30}$$

and

$$B_2 \approx \frac{5}{3}nkT\tau^2 \tag{31}$$

The cosmological viscosity η_c will be relevant when τ is of order one. Since for the Boltzmann dilute regime $\tau \sim 10^{-5}s - 10^{-8}s$, and for such values densities of the order 10^{-5}, $10^{-19}g/cm^3$ should prevail, corresponding roughly to the threshold of the matter domination era in the universe. Under these circumstances, the Burnett corrections which contain $\tau^2 \sim 10^{-16}s^2$ would require densities of the order of magnitude of $10^{-11}g/cm^3$, the density of matter right after nucleosynthesis, an era where radiation is still relevant [15]. Further study of these consequences will be reported later.

4 Discussion of the results

In this paper we have used the BGK model to describe the non-relativistic kinetic properties of a dilute gas in a FRW metric implying an isotropic and homogeneous universe. Two important results stem out of this analysis. In the first place the concept itself of transport coefficients is clarified. There existing no spatial gradients in the velocity, there will be no shear effects in the gas. Thus, ordinary viscosities, shear and volume, are not present. However, the contributions to the four dimensional covariant derivative of the velocity arising only from the geometric structure of space-time, embedded in the Christoffel symbols are present. They give rise to fluxes T^{μ}_{ν} arising from the dynamical character of the fluid. In the case here studied, the force turns out to be the expansion scalar θ. As we have shown elsewhere [14], there is a shear-like viscosity arising from geometry. Regretfully the nomenclature used in that paper is inappropriate, the coefficients then referred to are precisely η_c and its "shear-like" counterpart which should be called something else, not shear viscosity.

The second important result is that as Eq. (23)-(29) point out, there was a Burnett-like hydrodynamic regime in the evolution of the universe which reflects itself in the field equations themselves, as well as in the transport properties of matter. Nevertheless, it did so at times when matter was still coupled to radiation, so that it may me useful to add to this analysis the behavior of radiation. This subject is, at present, one of study.

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